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The Ziegler effect in a non-conservative mechanical system☆

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ABSTRACT

The destabilization of the stable equilibrium of a non-conservative system under the action of an infinitesimal linear viscous friction force is considered. In the case of low friction, the necessary and sufficient conditions for stability of a system with several degrees of freedom and, as a consequence, the conditions for the existence of the destabilization effect (Ziegler's effect) are obtained. Criteria for the stability of the equilibrium of a system with two degrees of freedom, in which the friction forces take arbitrary values, are constructed. The results of the investigation are applied to the problem of the stability of a two-link mechanism on a plane, and the stability regions and Ziegler's areas are constructed in the parameoter space of the problem.

1. Introduction

Consider a holonomic mechanical system with stationary ideal constraints under the action of potential and non-conservative positional forces, as well as dissipative viscous friction forces that depend linearly on the generalized velocities. Lagrange's equations for such a system have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q(q) - \frac{\partial \Phi}{\partial \dot{q}}; \quad \Phi = \frac{\varepsilon}{2}\left(\tilde{B}\dot{q}, \dot{q}\right) \tag{1.1}$$

Here $q = (q_1, \ldots, q_n)^T$ is the vector of the generalized coordinates, the kinetic energy T is a quadratic form in the generalized velocities, $Q(q)$ is the vector of generalized forces that corresponds to the non-conservative positional force, Φ is the dissipative Rayleigh function, which is quadratic in the velocities, and $\varepsilon > 0$ is the viscous friction coefficient. The dissipation is assumed to be complete; therefore, the matrix $\tilde{B}$ is positive definite. Without loss of generality, we will assume that $q = \dot{q} = 0$ is the isolated equilibrium of Eqs (1.1).

We will consider the problem of the influence of friction forces on the stability of the trivial equilibrium $q = \dot{q} = 0$. As is well known, when there are no friction forces and non-conservative positional forces, the stability of equilibrium is ensured by the condition that there is a potential-energy minimum at the equilibrium. In this case, an addition of dissipative forces with complete dissipation leads to asymptotic stability of the equilibrium (the Thomson–Tait–Chetayev theorem). However, the appearance of a non-conservative force $Q(q)$ can radically alter the behaviour of the system in the vicinity of the equilibrium. Investigating the stability of a double pendulum loaded with a follower force, Ziegler[1] arrived at the unexpected conclusion that the critical force F_* (for which a loss of stability of the equilibrium configuration of the system occurs) when there is no dissipation exceeds the critical force $F_{**}(\varepsilon b)$ of a system containing dissipative forces with an infinitesimal friction coefficient ε, including the limiting case in which $\varepsilon = 0$. This means that there is a region of variation of the absolute value of the follower force $F_* > F > F_{**}(0)$, in which the system is stable in a first approximation with no friction forces and this stability is destroyed by infinitesimal friction forces. We will call the presence of such regions in the parameter space the Ziegler effect, and the regions themselves will be called Ziegler's areas.

The influence of friction forces on the stability of the equilibrium of a holonomic system under the action of potential, gyroscopic, and non-conservative positional forces has been investigated.[2–5] It was shown in Ref. 2 that when dissipative forces with equal dissipation coefficients are added, an equilibrium that is stable in a first approximation becomes asymptotically stable (for the simplest results, see Ref. 6). However, one of the problems described in that paper, namely, the formulation of the necessary and sufficient conditions for the appearance of the Ziegler effect, has not heretofore been solved.

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Consider the equation of linear oscillations

$$A\ddot{q} + \varepsilon\tilde{B}\dot{q} + \tilde{C}q = 0 \tag{1.2}$$

where A is the positive definite matrix of kinetic energy coefficients, $\tilde{B}$ is the positive definite matrix of friction forces (the dissipation is complete), and $\tilde{C} = [\Pi_{q_j q_k}] - Q_q|_0$ is the matrix of positional forces, which consists of the symmetrical component $\tilde{C}_1$ and the skew-symmetric component $\tilde{C}_2$. It is assumed that $\Pi \in C^2(\Omega)$ and $Q \in C^1(\Omega; \mathbb{R}^n)$, where Ω is a region in $\mathbb{R}'$ that contains the origin of coordinates.

We introduce the normal coordinates x using the formula $q = Sx$, where S is a non-singular matrix that satisfies the conditions $S^TAS = I$ and $S^T\tilde{C}_1S = C$, in which C is a diagonal matrix.[7] Then Eq. (1.2) takes the form

$$\ddot{x} + \varepsilon B\dot{x} + Cx + Px = 0 \tag{1.3}$$

The positive definite matrix B is calculated using the formula $B = S^T\tilde{B}S$, and the skew-symmetric matrix P is calculated using the formula $P = S^T\tilde{C}_2S$.

The characteristic equation of system (1.3) has the form

$$\Delta(\lambda;\varepsilon) \equiv \det\left(\lambda^2 I + \varepsilon\lambda B + C + P\right) = 0 \tag{1.4}$$

When there are no friction forces ($\varepsilon = 0$), the linear equations of motion

$$\ddot{x} + Cx + Px = 0 \tag{1.5}$$

are reversible; therefore, asymptotic stability of the solution $x = 0$ is impossible, and stability only occurs when its eigenvalues are pure imaginary and semisimple.

Definition. Suppose the stationary solution $q = \dot{q} = 0$ of Eqs (1.1) is stable in a first approximation when $\varepsilon = 0$. We will say that the Ziegler effect (destabilization by friction forces) occurs in system (1.1) if the solution $q = \dot{q} = 0$ is Lyapunov unstable for any infinitesimal $\varepsilon > 0$.

2. Stability of a system in the case of small friction forces

We start out from the simple case of a system with two degrees of freedom ($n = 2$). The characteristic polynomial has the form

$$\Delta(\lambda;\varepsilon) \equiv \lambda^4 + \varepsilon \operatorname{tr} B\lambda^3 + \left(\operatorname{tr} C + \varepsilon^2 \det B\right)\lambda^2 + \varepsilon h\lambda + \det(C + P)$$

where

$$h = \operatorname{tr} B \operatorname{tr} C - \operatorname{tr}(BC) = c_1b_2 + c_2b_1$$

It is clear that $\operatorname{tr} B > 0$ by virtue of the fact that the symmetrical matrix B is positive definite.

Consider the auxiliary polynomial

$$\Delta_0(\omega) \equiv \Delta(i\omega;0) = \omega^4 - \operatorname{tr} C\omega^2 + \det(C + P)$$

It has the roots $\omega_2 > \omega_1 > 0$ if and only if the following equalities are satisfied:

$$\operatorname{tr} C > 0,\ \det(C + P) > 0,\ (\operatorname{tr} C)^2 - 4\det(C + P) > 0 \tag{2.1}$$

If inequalities (2.1) are satisfied, the solution $x = \dot{x} = 0$ of Eqs (1.4) will clearly be stable, while violation of these inequalities will result in instability by virtue of the appearance of roots of the characteristic equation with a positive real part. In the degenerate cases

$$\begin{aligned} &\operatorname{tr} C > 0,\ (\operatorname{tr} C)^2 - 4\det(C + P) = 0 \\ &\operatorname{tr} C \geq 0,\ \det(C + P) = 0 \end{aligned} \tag{2.2}$$

we have multiple, pure imaginary roots of the characteristic equation of system (1.4) (including multiple zero roots). We will not examine these cases.

The formula for determining the roots has the form

$$\omega_{1,2}^2 = \frac{1}{2}\left\{\operatorname{tr} C \mp \sqrt{(\operatorname{tr} C)^2 - 4\det(C + P)}\right\}$$

Theorem 1. Suppose inequalities (2.1), which ensure stability of the equilibrium of Eqs (1.1) in a first approximation, hold. If the parameter h satisfies the inequalities

$$\omega_1^2 \operatorname{tr} B \leq h \leq \omega_2^2 \operatorname{tr} B \tag{2.3}$$

there is a value $\varepsilon_1 > 0$, such that the equilibrium $q = \dot{q} = 0$ of Eqs (1.1) is asymptotically stable when $\varepsilon \in (0, \varepsilon_1]$, while instability occurs if

$$h < \omega_1^2 \operatorname{tr} B \text{ or } h > \omega_2^2 \operatorname{tr} B \tag{2.4}$$

when $\varepsilon \in (0, \varepsilon_1]$.

Suppose conditions (2.1) are violated. Then, apart from the degenerate cases (2.2), the equilibrium $q = \dot{q} = 0$ of Eqs (1.1) is unstable for $\varepsilon \in (0, \varepsilon_1]$.

Proof. Suppose λ_* is a root of the equation $\Delta(\lambda;\, 0) = 0$. The following alternatives are possible: $\lambda_* = \pm i\omega_2$ or $\lambda_* = \pm i\omega_1$. According to the implicit function theorem, if the derivative Δ_λ is non-zero at the point $(\lambda_*, 0)$ (i.e., the root λ_* is simple), there is an analytic function $\lambda = \lambda(\varepsilon)$, such that $\lambda(0) = \lambda_*$ and $\Delta(\lambda(\varepsilon);\varepsilon) = 0$ in the vicinity of $\varepsilon = 0$. It follows from the third inequality in conditions (2.1) that the root λ_* is simple.

In the expansion $\lambda(\varepsilon) = \lambda_* + \lambda^{(1)}\varepsilon + O(\varepsilon^2)$ the coefficient $\lambda^{(1)}$ is real:

$$\lambda^{(1)} = -\frac{\Delta_\varepsilon(\lambda_*;0)}{\Delta_\lambda(\lambda_*;0)} = \frac{h - \omega_*^2 \operatorname{tr} B}{4\omega_*^2 - 2\operatorname{tr} C}$$

The quantity in the denominator of the last expression can be either positive (if $\omega_* = \omega_2$) or negative (if $\omega_* = \omega_1$). The coefficient satisfies the inequality $\lambda^{(1)} < 0$ for both roots $\omega_* = \omega_1$ and $\omega_* = \omega_2$ if and only if

$$\omega_1^2 \operatorname{tr} B < h < \omega_2^2 \operatorname{tr} B$$

In this case $\operatorname{Re}\lambda(\varepsilon) < 0$ for both roots when ε is small, and the equilibrium of Eqs (1.1) is asymptotically stable. Conversely, if condition (2.4) is satisfied, then for a certain root $\lambda^{(1)} > 0$, $\operatorname{Re}\lambda(\varepsilon) > 0$, and instability develops.

The case of $\lambda^{(1)} = 0$ still needs to be investigated. It occurs when $h = \omega^2{}_* \operatorname{tr} B$. Substituting the expansion $\lambda(\varepsilon) = \lambda_* + \lambda^{(1)}\varepsilon + \lambda^{(2)}\varepsilon^2 + \lambda^{(3)}\varepsilon^3 + \ldots$ into the equation $\Delta(\lambda;\varepsilon) = 0$, we can write

$$\left(4\lambda_*^3\lambda^{(2)} + 2\lambda_*\lambda^{(2)}\operatorname{tr} C + \lambda_*^2 \det B\right)\varepsilon^2 +$$
$$+\left(4\lambda_*^3\lambda^{(3)} + 2\lambda_*^2\lambda^{(2)}\operatorname{tr} B + 2\lambda_*\lambda^{(3)}\operatorname{tr} C + h\lambda^{(2)}\right)\varepsilon^3 + \ldots = 0$$

After setting the coefficients in front of ε^2 and ε^3 equal to zero, we successively calculate $\lambda^{(2)}$ and $\lambda^{(3)}$. We obtain

$$\lambda^{(2)} = -\frac{\lambda_* \det B}{2\lambda_*^2 + 2\operatorname{tr} C} = \frac{\mp i\omega_* \det B}{-4\omega_*^2 + 2\operatorname{tr} C}$$

$$\lambda^{(3)} = \frac{-\lambda^{(2)}\left(2\lambda_*^2 \operatorname{tr} B + h\right)}{4\lambda_*^3 + 2\lambda_* \operatorname{tr} C} = -\frac{\omega_*^2 \det B \operatorname{tr} B}{\left(-4\omega_*^2 + 2\operatorname{tr} C\right)^2}$$

It can be seen that $\lambda^{(2)}$ is pure imaginary and that $\lambda^{(3)} < 0$; therefore, $\operatorname{Re}\lambda(\varepsilon) < 0$. Thus, the case of equality in relations (2.3) has been investigated.

We will assume that condition (2.1) is violated. This means that the characteristic polynomial has roots with a positive real part. Small friction forces do not influence the sign of the real part; therefore, the equilibrium is unstable. The theorem is proved.

Elementary calculations show that inequalities (2.1) and (2.4) are incompatible when $P = 0$. This means that the equilibrium destabilization effect cannot be caused by small friction forces when there are no non-conservative forces. Theorem 1 enables us to formulate the necessary and sufficient conditions for destabilization of the stable equilibrium of system (1.1) by small friction forces when there are non-conservative forces.

Corollary 1. Viscous friction forces of infinitesimal magnitude destabilize the equilibrium of system (1.1) with two degrees of freedom, which is stable in a first approximation, if and only if the following conditions are satisfied simultaneously:

1) the system is subjected to the action of non-conservative forces, i.e., $P \neq 0$,
2) the parameters of the system satisfy inequalities (2.4).

We will show that condition (2.3) is satisfied automatically for equal dissipation coefficients under condition (2.1). In fact, it follows from the equality $\Delta_0(\omega) = 0$ that $\omega_1^2 + \omega_2^2 = c_1 + c_2$. Therefore, inequality (2.3) can be rewritten as

$$2b_1\omega_1^2 \le b_1\left(\omega_1^2 + \omega_2^2\right) \le 2b_1\omega_2^2$$

which is satisfied, since $\omega_1 < \omega_2$.

We will examine the derived parameters

$$\beta = \frac{b_1}{\operatorname{tr} B}, \quad \theta = \frac{2\det P}{(c_1 - c_2)^2}$$

Then inequality (2.3) can be rewritten as $\theta < -4\beta^2 + 4\beta$. In Fig. 1 Ziegler's area is indicated by hatching.

We will now analyse the stability of the equilibrium for non-conservative systems with n degrees of freedom. The characteristic polynomial has the general form

$$\Delta(\lambda;\varepsilon) = \lambda^{2n} + \sum_{k=0}^{2n} \alpha_k \lambda^{2n-k} \tag{2.5}$$

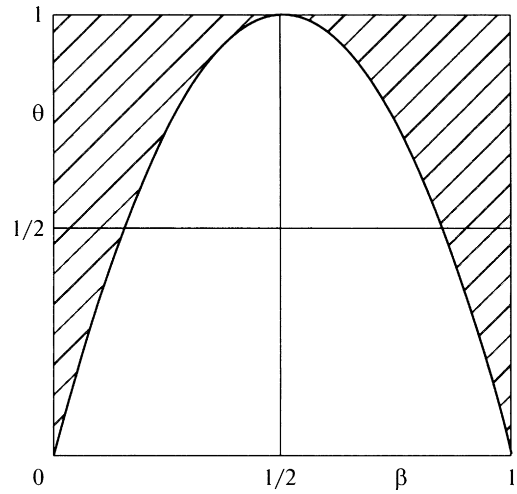

Fig. 1.

The coefficients α_k, which can be determined using Leverrier's algorithm,[8] are polynomials in ε, and

$$\alpha_{2j} = \alpha_{2j}^{(0)} + \alpha_{2j}^{(2)}\varepsilon^2 + \ldots,\ \ \alpha_{2j-1} = \alpha_{2j-1}^{(1)}\varepsilon + \ldots,\ \ j = 1,\ldots,n$$

We will consider the auxiliary polynomials

$$\Delta_0(\omega) = \Delta(i\omega;0) = \det\left(-\omega^2 I + C + P\right),\ \ \Delta_1(\omega) = -\sum_{j=1}^{n}\alpha_{2j-1}^{(1)}\left(-\omega^2\right)^{n-j}$$

$$\Delta_2(\omega) = \sum_{j=0}^{n-1} 2(n-j)\alpha_{2j}^{(0)}\left(-\omega^2\right)^{n-j-1},\ \ \alpha_0^{(0)} = 1$$

The point $x = \dot{x} = 0$ of Eqs (1.5) is stable if and only if the frequency equation $\Delta_0(\omega) = 0$ has only real roots. Suppose $\omega_n > \ldots > \omega_1 > 0$ are simple real roots of this equation.

The following theorem generalizes Theorem 1 and specifies the stability criteria of a system with n degrees of freedom with small friction forces.

Theorem 2. Suppose the polynomial Δ_0 has n real positive roots ω_l ($l = 1, \ldots, n$). If the coefficient $\alpha_{2n-1}^{(0)}$ satisfies the inequality

$$\min_s \Delta_1(\omega_{2s-1}) < \alpha_{2n-1}^{(0)} < \max_s \Delta_1(\omega_{2s}),\ \ s = 1,\ldots,[n/2], \tag{2.6}$$

there is a value $\varepsilon_1 > 0$, such that the equilibrium $q = \dot{q} = 0$ of Eqs (1.1) is asymptotically stable when $\varepsilon \in (0, \varepsilon_1]$, while instability occurs for

$$\alpha_{2n-1}^{(0)} < \min_s \Delta_1(\omega_{2s-1})\ \text{ or }\ \alpha_{2n-1}^{(0)} > \max_s \Delta_1(\omega_{2s}) \tag{2.7}$$

when $\varepsilon \in (0, \varepsilon_1]$.

We will assume that the polynomial Δ_0 has roots that do not lie on the real axis. Then, apart from the degenerate cases, where the frequency equation has multiple or zero roots, the equilibrium $q = \dot{q} = 0$ of Eqs (1.1) is unstable if $\varepsilon \in [0, \varepsilon_1]$.

Proof. We will examine how the root $\lambda_l^* = \pm i\omega_l$ behaves when there is a perturbation in system (1.3). According to the implicit function theorem, if the root λ_l^* is simple, there is a unique analytic function $\lambda(\varepsilon)$ in the vicinity of $\varepsilon = 0$ that is a solution of the problem $\Delta(\lambda;\varepsilon) = 0$, $\lambda(0) = \lambda_l^*$. In the expansion

$$\lambda(\varepsilon) = \lambda_l^* + \lambda_l^{(1)}\varepsilon + \lambda_l^{(2)}\varepsilon^2 + \ldots$$

we will determine the coefficient $\lambda_l^{(1)}$. For this purpose, we calculate the derivative $\Delta_\varepsilon(\lambda_l^*; 0)$. We have

$$\Delta_\varepsilon\left(\lambda_l^*;0\right) = \sum_{j=1}^{n}\alpha_{2j-1}^{(1)}(\pm i\omega_l)^{2(n-j)+1} = \pm i\omega_l\Delta_1(\omega_l)$$

Calculations show that

$$\Delta_\lambda\left(\lambda_l^*;0\right) = 2n(\pm i\omega_l)^{2n-1} + \sum_{j=1}^{n-1}2(n-j)\alpha_{2j}^{(0)}(\pm i\omega_l)^{2(n-j)-1} = \mp i\omega_l\Delta_2(\omega_l)$$

Finally, we write the expression

$$\lambda_l^{(1)} = -\frac{\Delta_\varepsilon\left(\lambda_l^*;0\right)}{\Delta_\lambda\left(\lambda_l^*;0\right)} = \frac{\Delta_1(\omega_l)}{\Delta_2(\omega_l)} \tag{2.8}$$

The coefficient $\lambda_l^{(1)}$ is real.

The destabilization in system (1.1) is associated with the appearance of a certain coefficient $\lambda_l^{(1)} > 0$. In this case Re $\lambda_l(\varepsilon) > 0$ in the vicinity of $\varepsilon = 0$. Conversely, if all the coefficients $\lambda_l^{(1)} < 0$, the perturbed system remains stable for small values of ε.

We will determine the sign of the coefficient $\lambda_l^{(1)}$. For this purpose, we calculate the sign of the denominator in expression (2.8) in terms of the sign of the derivative

$$\frac{d\Delta_0}{d\omega} = \frac{1}{\omega}\Delta_2(\omega_l)$$

The signs of $\frac{d\Delta_0}{d\omega}(\omega_l)$ alternate, but $\frac{d\Delta_0}{d\omega}(\omega_l) < 0$. This is attributed to the fact that the highest-order coefficient in the polynomial Δ_0 is equal to $(-1)^n$.

The derivative $\frac{d\Delta_0}{d\omega}(\omega_l)$ differs from the expression in the denominator on the right-hand side of equality (2.8) by the coefficient ω^{-1}. Taking all the above into account, we conclude that the condition $\lambda_l^{(1)} < 0$ ($l = 1, \ldots, n$) is equivalent to the condition

$$(-1)^{l-1}\Delta_1(\omega_l) > 0, \quad l = 1,\ldots,n \tag{2.9}$$

Conversely, destabilization occurs in the system when there is a value $l \in \{1, \ldots, n\}$, such that the inverse inequality to inequality (2.9) is satisfied. It is easy to see that condition (2.9) is equivalent to condition (2.6) and that the inverse inequality is equivalent to condition (2.7).

If the polynomial Δ_0 has roots that do not lie on the real axis, one of the characteristic roots will have a positive real part; therefore, the trivial equilibrium of system (1.1) will be unstable. Small friction forces do not alter this conclusion.

Using Leverrier's algorithm, we write the following algorithm for calculating the coefficients $\alpha_k^{(0)}$ and $\alpha_k^{(1)}$:

$$\alpha_1^{(0)} = 0, \quad \alpha_1^{(1)} = \operatorname{tr} B,\ C_0^{(1)} = 0, \quad C_1^{(1)} = 0,\ C_1^{(1)} = -B + I\operatorname{tr} B$$
$$\alpha_k^{(0)} = \frac{2}{k}\operatorname{tr}\left\{(C+P)C_{k-2}^{(0)}\right\}, \quad \alpha_k^{(1)} = \frac{1}{k}\operatorname{tr}\left\{BC_{k-1}^{(0)}\right\} + \frac{2}{k}\operatorname{tr}\left\{(C+P)C_{k-2}^{(1)}\right\}$$
$$C_k^{(0)} = -(C+P)C_{k-2}^{(0)} + \alpha_k^{(0)}I, \quad C_k^{(1)} = -BC_{k-1}^{(0)} - (C+P)C_{k-2}^{(1)} + \alpha_k^{(1)}; \quad k = 2,\ldots,2n-2$$
$$\alpha_{2n-1}^{(1)} = BC_{2n-2}^{(0)} + (C+P)C_{2n-3}^{(1)}, \quad \alpha_{2n}^{(0)}I = (C+P)C_{2n-2}^{(0)}$$

3. The Influence of large friction forces

Up to this point we have considered only small viscous friction forces in non-conservative system (1.1). From the published results[4] it can be concluded that viscous friction forces with sufficiently large coefficients can have a stabilizing influence on the equilibrium of the system. We will show that, under certain conditions, in a system with two degrees of freedom there is a large value of the viscous friction coefficient $\varepsilon_* > 0$, which is such that the equilibrium of Eqs (1.1) is asymptotically stable for $\varepsilon > \varepsilon_*$ and unstable for $\varepsilon < \varepsilon_*$. It is clear that such a statement of the problem is correct if the equilibrium of the perturbed system is unstable or if destabilization by small friction forces occurs in the system. We will call ε_* the critical value.

When $\varepsilon = \varepsilon_*$, the characteristic equation will clearly have the pure imaginary root $i\omega_*$. Hence it is easy to determine ε_* and ω_*. In fact, separating the real and imaginary parts in the complex equation

$$\Delta\left(i\omega_*;\varepsilon\right) \equiv \omega_*^4 - \left(\operatorname{tr} C + \varepsilon^2 \det B\right)\omega_*^2 + \det(C+P) + \varepsilon i\omega_*\left(-\omega_*^2 \operatorname{tr} B + h\right) = 0$$

we obtain the two equalities

$$-\omega_*^2 \operatorname{tr} B + h = 0,\ \omega_*^4 - \left(\operatorname{tr} C + \varepsilon^2 \det B\right)\omega_*^2 + \det(C+P) = 0$$

We assume that $\det(C+P) \neq 0$. If we exclude the degenerate case $\varepsilon = \varepsilon_*$, we find

$$\omega_*^2 = \frac{h}{\operatorname{tr} B}, \quad \varepsilon_*^2 = \frac{1}{h\det B}\left\{\frac{1}{\operatorname{tr} B}h^2 - h\operatorname{tr} C + \operatorname{tr} B\det(C+P)\right\} \tag{3.1}$$

Hence it follows that to solve the problem in question we must require that the following inequalities are satisfied

$$h > 0, \quad \varphi(h) > 0$$

where

$$\varphi(u) = \frac{1}{\operatorname{tr} B}u^2 - u\operatorname{tr} C + \operatorname{tr} B\det(C+P)$$

However, these conditions are not necessarily sufficient, because the fact that the characteristic equation has the two imaginary roots $\pm i\omega_*$ indicates that the remaining two roots maintain the sign of the real part when $\varepsilon = \varepsilon_*$, therefore, their influence is significant.

Lemma. Apart from relations of the equality type, the necessary and sufficient condition for the existence of a critical value of ε_* is that one of the following conditions holds

$$(\mathrm{tr}\,C)^2 - 4\det(C+P) < 0, \quad h > 0 \tag{3.2}$$

$$\mathrm{tr}\,C < 0, \quad (\mathrm{tr}\,C)^2 - 4\det(C+P) > 0, \quad \det(C+P) > 0, \quad h > 0 \tag{3.3}$$

$$\mathrm{tr}\,C > 0, \quad (\mathrm{tr}\,C)^2 - 4\det(C+P) > 0, \quad \det(C+P) > 0, \quad h \in \left(0, \omega_1^2 \mathrm{tr}\,B\right) \cup \left(\omega_2^2 \mathrm{tr}\,B, \infty\right) \tag{3.4}$$

The value of ε_* is specified by formula (3.1).

Proof. We note that if the discriminant of the equation $\varphi(u) = 0$

$$D = (\mathrm{tr}\,C)^2 - 4\det(C+P)$$

is less than zero (the first condition in (3.2)), then $\varphi(u) > 0$ for any u, and, in particular, $\varphi(h) > 0$ for $h > 0$.

Consider the case when $D > 0$. If the first three inequalities in (3.3) are satisfied, the equation $\varphi(u) = 0$ has two real negative roots; therefore, $\varphi(h) > 0$ for any $h > 0$. If inequalities (3.4) are satisfied, then $\varphi(h) > 0$ if and only if $h < u_1$ or $h > u_2$, where $u_{1,2}$ are roots of the equation $\varphi(u) = 0$ and $u_1 < u_2$. Since $u_{1,2} = \omega_{1,2}^2$ tr B, we arrive at condition (2.4).

We will now ascertain the conditions under which the parameter ε_* will be critical. Consider the Hurwitz matrix for the characteristic equation

$$\begin{Vmatrix} \varepsilon\,\mathrm{tr}\,B & \varepsilon h & 0 & 0 \\ 1 & \mathrm{tr}\,C + \varepsilon^2 \det B & \det(C+P) & 0 \\ 0 & \varepsilon\,\mathrm{tr}\,B & \varepsilon h & 0 \\ 0 & 1 & \mathrm{tr}\,C + \varepsilon^2 \det B & \det(C+P) \end{Vmatrix} \tag{3.5}$$

The Routh–Hurwitz criterion for the characteristic equation not having roots with a non-negative real part is the system of inequalities $\delta_j > 0$, where δ_j is the j-th corner minor of matrix (3.5). Since the highest-order coefficient of the polynomial Δ is positive, these inequalities are equivalent to the following: $\alpha_j > 0$, $\delta_3 > 0$, where α_j is the coefficient of λ^{4-j} in the characteristic polynomial. Since B is a positive definite matrix, $\alpha_1 > 0$. We obtain

$$\delta_3 = -\varepsilon^2 (\mathrm{tr}\,B)^2 \det(C+P) + \varepsilon^2 h\,\mathrm{tr}\,B\left(\mathrm{tr}\,C + \varepsilon^2 \det B\right) - \varepsilon^2 h^2$$

Reducing the inequality $\delta_3 > 0$ by ε^2 and solving it for ε, we arrive at the inequality $\varepsilon > \varepsilon_*$. The system of inequalities $\alpha_j > 0$ is equivalent to two inequalities: $h > 0$ and det $(C+P) > 0$. The latter inequality also follows from inequalities (3.2). When $\varepsilon > \varepsilon_*$, the inequality tr $C + \varepsilon^2$ det $B > 0$ is satisfied, as was established by a direct calculation.

If $0 < \varepsilon < \varepsilon_*$, then $\delta_3 > 0$, and the equilibrium of Eqs (1.1) is unstable.

In the remaining cases, in which the parameters of the system do not satisfy the conditions of the lemma, we have the following. Either the critical value ε_* does not exist because the conditions of the Hurwitz theorem (det $(C+P) < 0$) are violated for any values of the parameter ε and, therefore, the equilibrium is unstable regardless of magnitude of the friction forces, or $h < 0$ (equalities (3.1) become meaningless). A case of asymptotic stability is also possible for small friction forces (the first three inequalities in (3.4) hold, but h satisfies conditions (2.3)), and then the problem of finding ε_* becomes meaningless.

Thus, combining the assertions of Theorem 1 and the lemma, we obtain the following equilibrium stability criteria in a first approximation for the case of the system with two degrees of freedom when the friction forces take arbitrary values.

Theorem 3. I. Suppose equilibrium stability conditions (2.1) for Eqs (1.1) in a first approximation when there are no friction forces are satisfied. Then, if the parameter h satisfies the inequalities

$$0 < h < \omega_1^2\, tr\, B\, or\, h > \omega_2^2\, tr\, B$$

the trivial equilibrium of the system is unstable for $\varepsilon \in (0, \varepsilon_*)$ and asymptotically stable for $\varepsilon \in (\varepsilon_*, \infty)$. If the parameter h satisfies the inequalities

$$\omega_1^2 \mathrm{tr}\,B \le h \le \omega_2^2 \mathrm{tr}\,B,$$

the equilibrium is asymptotically stable for any ε. If $h < 0$, the equilibrium is unstable for any ε.

II. Suppose conditions (2.1) are violated, i.e., the equilibrium of the system is unstable in a first approximation when there are no friction forces. The following occurs: if the inequalities

$$(\mathrm{tr}\,C)^2 - 4\det(C+P) < 0, \quad h > 0$$

or

$$\mathrm{tr}\,C < 0, \quad (\mathrm{tr}\,C)^2 - 4\det(C+P) > 0, \quad \det(C+P) > 0, \quad h > 0$$

are satisfied, the equilibrium is unstable for $\varepsilon \in (0, \varepsilon_*)$ and asymptotically stable for $\varepsilon \in (\varepsilon_*,\infty)$. Otherwise, the equilibrium is unstable for any values of ε.

4. A two-link rod system

As an applied problem that illustrates the Ziegler effect, we will consider the problem of the stability of the equilibrium of a two-link mechanism in a horizontal plane. The non-conservative character of this system is associated with the action of the follower force on the free end of the second link. Follower force makes a constant angle with the axes of the bodies to which it is applied. This problem was previously considered in Refs. 9 and 10 and it generalizes the Ziegler problem.

Thus, we will examine a two-link system that lies on a horizontal plane and consists of heavy uniform rods AB and BC, whose masses are equal to m (Fig. 2). The rods are joined by spiral springs with coefficient of elasticity c. The follower force F is applied to the free end of the rod BC and makes a constant angle α with BC. Viscous friction with coefficient b at hinges A and B.

We introduce the generalized coordinates φ_1 and φ_2. The system investigated is a holonomic system with ideal stationary constraints and the following active forces: the elastic forces at the hinges A and B, the friction forces at these hinges and the follower force F. We write the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\varphi_j}-\frac{\partial T}{\partial\varphi_j}=-\frac{\partial\Pi}{\partial\varphi_j}-\frac{\partial\Phi}{\partial\dot\varphi_j}+Q_j,\quad j=1,2$$

Here T is the kinetic energy of the system, Π is the potential energy, Φ is the dissipative Rayleigh function, and Q_j is a generalized force that corresponds to the follower force (a non-conservative positional force). We write the expressions for T, Π, Φ and Q_j

$$T=\frac{m}{6}\left[4l^2\dot\varphi_1^2+3l^2\dot\varphi_1\dot\varphi_2\cos(\varphi_2-\varphi_1)+l^2\dot\varphi_2^2\right],\quad \Pi=\frac{c}{2}\left[\varphi_1^2+(\varphi_2-\varphi_1)^2\right]$$

$$\Phi=\frac{b}{2}\left[\dot\varphi_1^2+(\dot\varphi_2-\dot\varphi_1)^2\right],\quad Q_1=-Fl\sin(\varphi_2-\varphi_1-\alpha),\quad Q_2=Fl\sin\alpha$$

Lagrange's equations are written as follows:

$$\begin{aligned}&\frac{4}{3}ml^2\ddot\varphi_1+\frac{1}{2}ml^2\cos(\varphi_2-\varphi_1)\ddot\varphi_2-\frac{1}{2}ml^2\sin(\varphi_2-\varphi_1)\dot\varphi_2^2=\\&=c(\varphi_2-2\varphi_1)+b(\dot\varphi_2-2\dot\varphi_1)-Fl\sin(\varphi_2-\varphi_1-\alpha)\\&\frac{1}{2}ml^2\cos(\varphi_2-\varphi_1)\ddot\varphi_1+\frac{1}{3}ml^2\ddot\varphi_2+\frac{1}{2}ml^2\sin(\varphi_2-\varphi_1)\dot\varphi_1^2=\\&=c(\varphi_1-\varphi_2)+b(\dot\varphi_1-\dot\varphi_2)+Fl\sin\alpha\end{aligned}\tag{4.1}$$

The equilibrium positions φ_1^* and φ_2^* of the system under consideration are specified by the system of equations

$$\begin{aligned}&\varphi_2-2\varphi_1=\gamma\sin(\gamma\sin\alpha-\alpha)\\&\varphi_1-\varphi_2=-\gamma\sin\alpha\end{aligned}\tag{4.2}$$

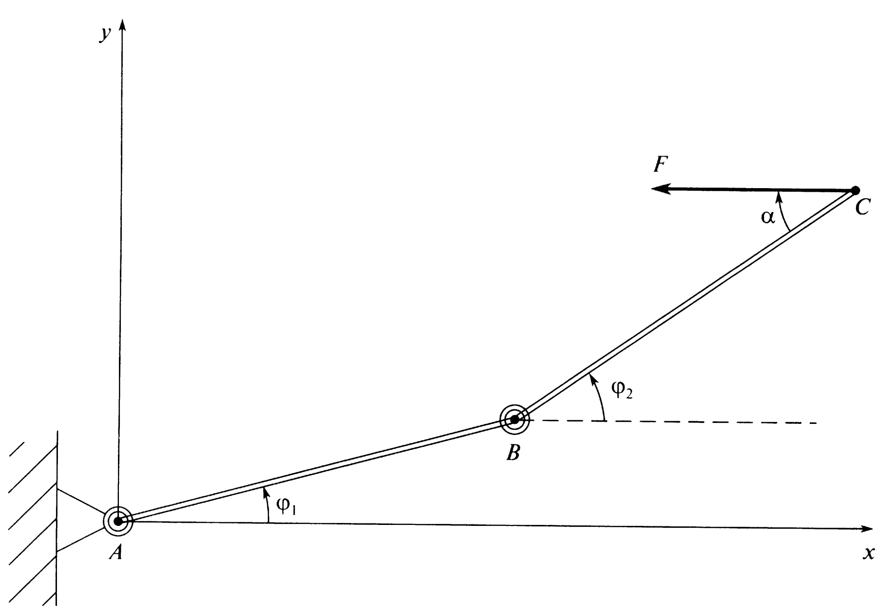

Fig. 2.

where $\gamma = Flc^{-1}$ is a dimensionless parameter. Solving system (4.2), we find

$$\varphi_1^* = \gamma\sin\alpha - \gamma\sin(\gamma\sin\alpha - \alpha)$$
$$\varphi_2^* = 2\gamma\sin\alpha - \gamma\sin(\gamma\sin\alpha - \alpha)$$

When $\alpha = 0$, we have $\varphi^*_1 = \varphi^*_2 = 0$ (the case considered by Ziegler). The mechanical system under consideration has a single equilibrium. We will investigate this equilibrium for stability.

We write the equations of perturbed motion in the vicinity of the equilibrium. We introduce the perturbations

$$\varphi_1 = \varphi_1^* + \beta_1,\quad \varphi_2 = \varphi_2^* + \beta_2$$

and make the equations of motion dimensionless by treating 1 radian as the unit of measuremant of the angle variables and the characteristic value $T_* = \sqrt{ml^2c^{-1}}$ as the unit of measurement of time. The equations of perturbed motion take the form

$$\begin{aligned}
&\frac{4}{3}\ddot\beta_1 + \frac{1}{2}\cos(\beta_2 - \beta_1 + \gamma\sin\alpha)\ddot\beta_2 - \frac{1}{2}\sin(\beta_2 - \beta_1 + \gamma\sin\alpha)\dot\beta_2^2 = \\
&= (\beta_2 - 2\beta_1 + \gamma\sin(\gamma\sin\alpha - \alpha)) + \varepsilon(\dot\beta_2 - 2\dot\beta_1) - \\
&- \gamma\sin(\beta_2 - \beta_1 - \alpha + \gamma\sin\alpha) \\
&\frac{1}{2}\cos(\beta_2 - \beta_1 + \gamma\sin\alpha)\ddot\beta_1 + \frac{1}{3}\ddot\beta_2 + \frac{1}{2}\sin(\beta_2 - \beta_1 + \gamma\sin\alpha)\dot\beta_1^2 = \\
&= (\beta_1 - \beta_2) + \varepsilon(\dot\beta_1 - \dot\beta_2)
\end{aligned} \tag{4.3}$$

Here $\varepsilon = b/(l\sqrt{mc})$, and a dot now denotes a derivative with respect to dimensionless time.

We will obtain the system for Eqs (4.3) in the first approximation. It can be written in vector-matrix form as follows:

$$A\ddot\beta + \varepsilon B\dot\beta + C\beta = 0,\quad \beta = (\beta_1, \beta_2)^T \tag{4.4}$$

where

$$A = \begin{Vmatrix} \frac{4}{3} & \frac{1}{2}\cos(\gamma\sin\alpha) \\ \frac{1}{2}\cos(\gamma\sin\alpha) & \frac{1}{3} \end{Vmatrix},\quad B = \begin{Vmatrix} 2 & -1 \\ 1 & 1 \end{Vmatrix}$$

$$C = \begin{Vmatrix} 2 - \gamma\cos(\gamma\sin\alpha - \alpha) & \gamma\cos(\gamma\sin\alpha - \alpha) - 1 \\ -1 & 1 \end{Vmatrix}$$

The stability of the solution $\beta = 0$, $\dot\beta = 0$ of Eq. (4.4) for $\varepsilon = 0$ and small non-zero values of ε must be investigated. We present the corresponding characteristic equation

$$\lambda^4\det A + \varepsilon(2 + \cos(\gamma\sin\alpha))\lambda^3 + \kappa\lambda^2 + 2\varepsilon\lambda + 1 = 0 \tag{4.5}$$

Here

$$\det A = \frac{4}{9} - \frac{1}{4}\cos^2(\gamma\sin\alpha)$$

$$\kappa = -\gamma\cos(\gamma\sin\alpha - \alpha)\left(\frac{1}{2}\cos(\gamma\sin\alpha) + \frac{1}{3}\right) + \cos(\gamma\sin\alpha) + 2 + \varepsilon^2$$

We write the Routh–Hurwitz conditions for the case in when Eq. (4.5) has no roots with a non-negative real part

$$\begin{aligned}
&\delta_1 = \varepsilon(2 + \cos(\gamma\sin\alpha)) > 0 \\
&\delta_2 = \delta_1\kappa - 2\varepsilon\det A > 0 \\
&\delta_3 = 2\varepsilon\delta_2 - \delta_1^2 > 0
\end{aligned} \tag{4.6}$$

The δ_k are corner minors of the matrix

$$\begin{Vmatrix} \varepsilon(2 + \cos(\gamma\sin\alpha)) & 2\varepsilon & 0 & 0 \\ \det A & \kappa & 1 & 0 \\ 0 & \varepsilon(2 + \cos(\gamma\sin\alpha)) & 2\varepsilon & 0 \\ 0 & 1 & \kappa & 1 \end{Vmatrix}$$

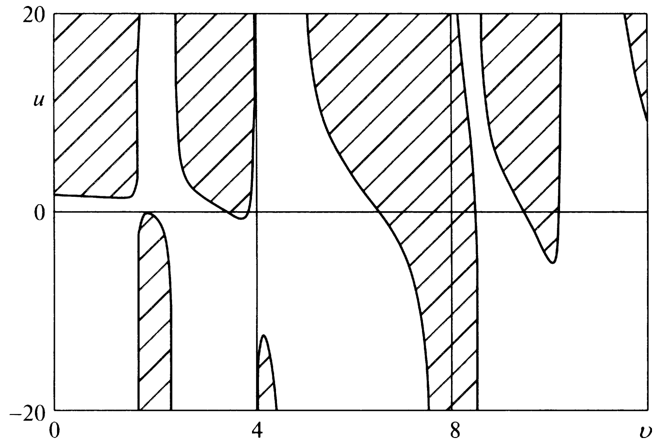

Fig. 3.

The minor $\delta_4=\delta_3$; therefore, the condition $\delta_4>0$ is not included in the list of conditions in (4.6). It is also obvious that $\delta_1>0$ for any α, $\gamma>0$. Inequalities (4.6) are equivalent to the inequality $\delta_3>0$, which can be written in the form

$$\varepsilon^2+1+\frac{1}{2}\cos(\gamma\sin\alpha)-$$
$$-\gamma\cos(\gamma\sin\alpha-\alpha)\left(\frac{1}{2}\cos(\gamma\sin\alpha)+\frac{1}{3}\right)>\left[\frac{8}{9}-\frac{1}{2}\cos^2(\gamma\sin\alpha)\right][2+\cos(\gamma\sin\alpha)]^{-1} \tag{4.7}$$

Thus, inequality (4.7) defines the stability region of the equilibrium of Eqs (4.4). We will construct it for $\varepsilon\downarrow 0$. For this purpose we introduce the new parameters u and υ, γ and α by the expressions

$$u=\gamma\sin\alpha,\quad \upsilon=\gamma\cos\alpha$$

and we rewrite inequality (4.7) in the form

$$\upsilon\left(\cos u+\frac{2}{3}\right)\cos u<2+\cos u-u\left(\cos u+\frac{2}{3}\right)\sin u+\frac{\cos^2 u-16/9}{2+\cos u} \tag{4.8}$$

We will consider the function $f(u)=(\cos u+2/3)\cos u$ and find the sets on which it takes values of the same sign. Suppose $0\le u\le 2\pi$. Then

$$f(u)>0\quad\text{for}\quad u\in\left[0,\frac{\pi}{2}\right)\cup\left(\pi-\arccos\frac{2}{3},\pi+\arccos\frac{2}{3}\right)\cup\left(\frac{3\pi}{2},2\pi\right];$$
$$f(u)<0\quad\text{for}\quad u\in\left(\frac{\pi}{2},\pi-\arccos\frac{2}{3}\right)\cup\left(\pi+\arccos\frac{2}{3},\frac{3\pi}{2}\right) \tag{4.9}$$

We will denote the right-hand side of inequality (4.8) by $g(u)$. Then, for the values of u for which $f(u)>0$ ($f(u)<0$), inequality (4.8) is written as $\upsilon<g(u)/f(u)$ ($\upsilon>g(u)/f(u)$). Region (4.8) is shown in Fig. 3.

The hatched areas correspond to points of instability, and the clear areas correspond to points of stability, i.e., solutions of inequality (4.8). It is seen that the stability region consists of a denumerable number of continuous components.

We will transpose the stability region into the strips (γ,α). For this purpose we construct the curves

$$\gamma(u)=\sqrt{u^2+\frac{g(u)^2}{f(u)^2}},\quad \alpha(u)=-\operatorname{arctg}\frac{g(u)}{uf(u)}+\frac{\pi}{2}$$

The parameter u runs through one of the intervals appearing in formula (4.9) with displacement by $2\pi k$ ($k=0, 1, 2\ldots$). The union of the denumerable number of these curves is the boundary of the stability region in the parameters (γ,α) (Fig. 4).

We will compare stability region (4.8) and the stability region of Eqs (4.4) when $\varepsilon=0$. It was shown in Ref. 10 that the stability region (in a first approximation) of a system without dissipation is specified by the inequality

$$\frac{1}{6}(2+3\cos(\gamma\sin\alpha))(2-\gamma\cos(\gamma\sin\alpha-\alpha))>-\frac{4}{3}+\sqrt{\frac{16}{9}-\cos^2(\gamma\sin\alpha)} \tag{4.10}$$

Using the parameters u and υ, we can rewrite this inequality in the form

$$\upsilon\cos u\left(\cos u+\frac{2}{3}\right)<4-2\sqrt{\frac{16}{9}-\cos^2 u}-\frac{2}{3}u\sin u+2\cos u-u\sin u\cos u \tag{4.11}$$

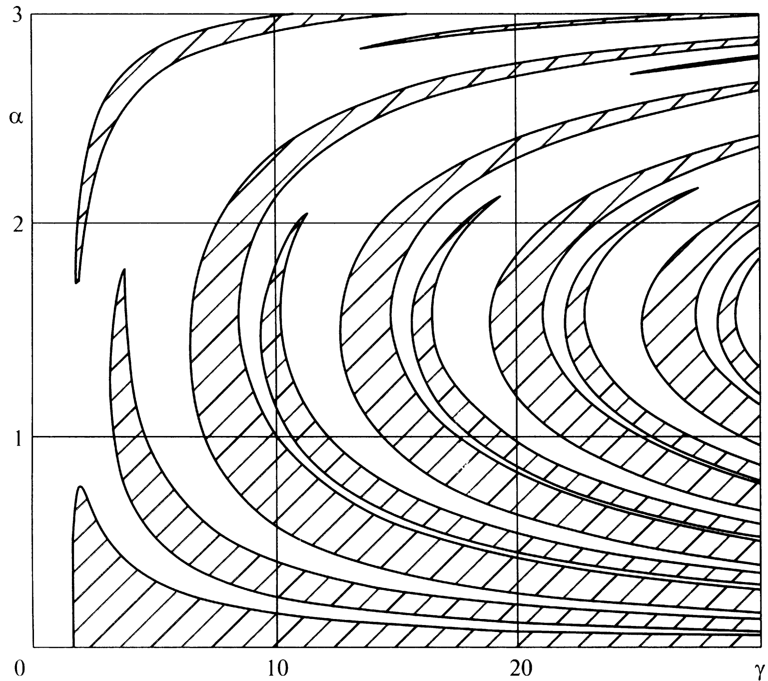

Fig. 4.

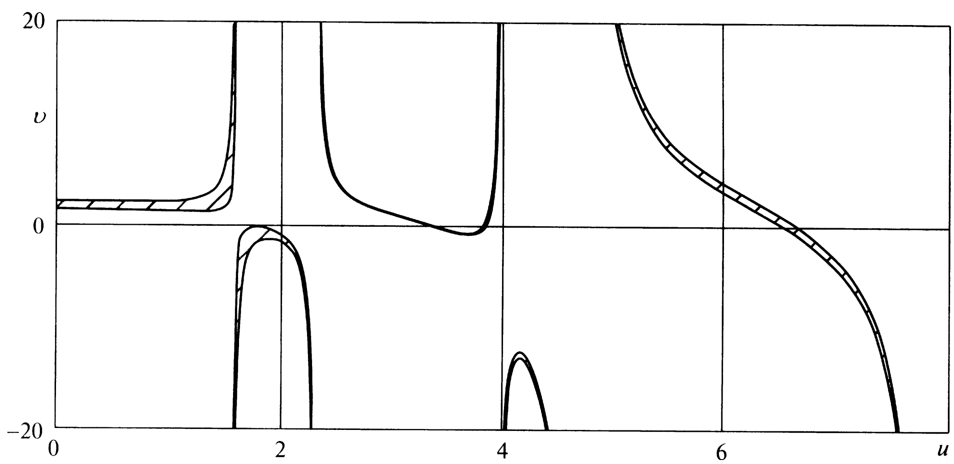

Fig. 5.

Elementary calculations show that inequality (4.10) is a consequence of inequality (4.8). This means that the stability region without dissipation includes the stability region of the system with dissipation and is wider than the latter. The point (u,υ) belongs to Ziegler's area if and only if inequality (4.10) is satisfied, but inequality (4.8) is not satisfied (Fig. 5).

We will construct the region of stabilization of the equilibrium of system (4.11) by large friction forces. We refer to the lemma. Conditions (3.4) correspond to Ziegler's areas. It can be shown that inequalities (3.2) and (3.3) are incompatible for the problem under consideration. Thus, the stabilization region is identical with Ziegler's areas.

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